

ENDOMORPHISMS OF THE LATTICE OF EPIGROUP VARIETIES

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ABSTRACT. We examine varieties of epigroups as unary semigroups, that is semigroups equipped with an additional unary operation of pseudoinversion. The article contains two main results. The first of them indicates a countably infinite family of injective endomorphisms of the lattice of all epigroup varieties. An epigroup variety is said to be a *variety of finite degree* if all its nilsemigroups are nilpotent. The second result of the article provides a characterization of epigroup varieties of finite degree in a language of identities and in terms of minimal forbidden subvarieties. Note that the first result is essentially used in the proof of the second one.

1. INTRODUCTION AND SUMMARY

A semigroup S is called an *epigroup* if some power of each element of S lies in a subgroup of S . The class of epigroups is quite wide. It includes, in particular, all *completely regular* semigroups (i. e., unions of groups) and all *periodic* semigroups (i. e., semigroups in which every element has an idempotent power). Epigroups have been intensively studied in the literature under different names since the end of 1950's. An overview of results obtained here is given in the fundamental work by L. N. Shevrin [10] and his survey [11].

It is natural to consider epigroups as *unary semigroups*, i. e., semigroups equipped with an additional unary operation. This operation is defined by the following way. If S is an epigroup and $a \in S$ then some power of a lies in a maximal subgroup of S . We denote this subgroup by G_a . The unit element of G_a is denoted by a^ω . It is well known (see [10], for instance) that the element a^ω is well defined and $aa^\omega = a^\omega a \in G_a$. We denote the element inverse to aa^ω in G_a by \bar{a} . The map $a \mapsto \bar{a}$ is the unary operation on S mentioned above. The element \bar{a} is called *pseudoinverse* to a . Throughout this article we consider epigroups as algebras with two operations, namely multiplication and pseudoinversion. In particular, this allows us to speak about varieties of epigroups as algebras with these two operations. An investigation of epigroups in the framework of the theory of varieties was promoted by L. N. Shevrin in

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the mentioned article [10]. An overview of first results obtained here may be found in [13, Section 2].

It is well known (see [10, 11], for instance) that the class of all epigroups is not a variety. In other words, the variety of unary semigroups generated by this class contains not only epigroups. Denote this variety by $\mathcal{EP}\mathcal{I}$. We note that an identity basis of the variety $\mathcal{EP}\mathcal{I}$ is known. This result was announced in 2000 by Zhil'tsov [18], and its proof was rediscovered recently by Mikhailova [5] (some related results can be found in [2]).

If \mathcal{V} is a semigroup [epigroup] variety then we denote by $\overleftarrow{\mathcal{V}}$ the variety consisting of semigroups [epigroups] dual (that is, antiisomorphic) to semigroups [epigroups] from \mathcal{V} . It is evident that the mapping δ of the lattice of all semigroup varieties **SEM** [the lattice of all epigroup varieties **EPI**] into itself given by the rule $\delta(\mathcal{V}) = \overleftarrow{\mathcal{V}}$ for every variety \mathcal{V} is an automorphism of this lattice. The question whether there are non-trivial automorphisms of the lattice **SEM** [the lattice **EPI**] different from δ still open so far. We notice that there exist infinitely many non-trivial injective endomorphisms of the lattice **SEM**. Namely, let \mathcal{V} be a semigroup variety given by the identities $\{u_i = v_i \mid i \in I\}$, let m and n be non-negative integers and $x_1, \dots, x_m, y_1, \dots, y_n$ letters that do not occur in the words u_i and v_i for all $i \in I$. Let $\mathcal{V}^{m,n}$ be the semigroup variety given by the identities

$$\{x_1 \cdots x_m u_i y_1 \cdots y_n = x_1 \cdots x_m v_i y_1 \cdots y_n \mid i \in I\}.$$

It is verified in [4] that the variety $\mathcal{V}^{m,n}$ does not depend on the choice of an identity basis of the variety \mathcal{V} and the mapping $\mathcal{V} \mapsto \mathcal{V}^{m,n}$ is an injective endomorphism of the lattice **SEM**. The first main result of this article is an epigroup analog of the mentioned fact.

In order to formulate this result, we need some definitions and notation. We denote by F the free unary semigroup. The unary operation on F will be denoted by $\bar{}$. Elements of F are called *unary words* or simply *words*. Let Σ be a system of identities written in the language of unary semigroups (that is, the language that consists of one associative binary operation and one unary operation). Then K_Σ stands for the class of all epigroups satisfying Σ (here we treat the unary operation from our language as the pseudoinversion). The class K_Σ is not obliged to be a variety because it maybe not closed under taking of infinite Cartesian products (see [11, Subsection 2.3] or Example 2.14 below, for instance). Note that this class is a generalized variety in the sense of [1]. A complete classification of identity system Σ such that K_Σ is a variety is provided by Proposition 2.15 below. If Σ has this property then we will write $\mathbf{V}[\Sigma]$ alongwith (and in the same sense as) K_Σ . We denote a variety of unary semigroups that satisfy identity system Σ by $\text{var } \Sigma$. Denote the set of all identities that hold in any epigroup by Δ . Thus $\mathcal{EP}\mathcal{I} = \text{var } \Delta$. Let $\text{var}_E \Sigma = \mathcal{EP}\mathcal{I} \wedge \text{var } \Sigma = \text{var } (\Sigma \cup \Delta)$ (here the symbol \wedge denotes the meet of varieties). Clearly, if the class K_Σ is not a variety then $\text{var}_E \Sigma$ contains some unary semigroups that are not epigroups. Moreover, the classes K_Σ and $\text{var}_E \Sigma$ may differ even whenever K_Σ is a variety (see Example 2.16 below). The first main result of the article is the following

Theorem 1.1. *Let $\mathcal{V} = \text{var}_E\{u_i = v_i \mid i \in I\}$ be an epigroup variety, let m and n be non-negative numbers, and $x_1, \dots, x_m, y_1, \dots, y_n$ letters that do not occur in the words v_i and w_i for all $i \in I$. Put*

$$\mathcal{V}^{m,n} = \text{var}_E\{x_1 \cdots x_m u_i y_1 \cdots y_n = x_1 \cdots x_m v_i y_1 \cdots y_n \mid i \in I\}.$$

*Then the variety $\mathcal{V}^{m,n}$ is an epigroup variety and the map $\mathcal{V} \mapsto \mathcal{V}^{m,n}$ is an injective endomorphism of the lattice **EPI**.*

An examination of semigroup varieties shows that properties of a variety are depended in an essential degree by properties of nilsemigroups belonging to the variety. More precisely, we have in mind the question, whether a variety contains non-nilpotent nilsemigroups; if it is not the case then what about nilpotency degrees of nilsemigroups in a variety? This gives natural the following definitions. A semigroup variety \mathcal{V} is called a *variety of finite degree* if all nilsemigroups in \mathcal{V} are nilpotent. If \mathcal{V} has a finite degree then it is said to be a *variety of degree n* if nilpotency degrees of all nilsemigroups in \mathcal{V} does not exceed n and n is the least number with such a property. Semigroup varieties of finite degree and some natural subclasses of this class of varieties were investigated in [6, 9, 14–16] and other articles (see also Section 8 in the survey [12]).

It is well known and may be easily verified that, in a periodic semigroup, pseudoinversion may be written by using of multiplication only. Indeed, if an epigroup satisfies the identity

$$(1.1) \quad x^p = x^{p+q}$$

for some natural numbers p and q then the identity

$$(1.2) \quad \overline{x} = x^{(p+1)q-1}$$

holds in this epigroup. If $p > 1$ then the simpler identity

$$(1.3) \quad \overline{x} = x^{pq-1}$$

is valid. This means that periodic varieties of epigroups may be identified with periodic varieties of semigroups. Semigroup varieties of finite degree are periodic, whence they may be considered as epigroup varieties. It seems to be natural to expand the notions of varieties of finite degree or of degree n to all epigroup varieties. Definitions of epigroup varieties of finite degree or degree n repeat literally definitions of the same notions for semigroup varieties.

In [9, Theorem 2], semigroup varieties of finite degree were characterized in several ways. In particular, it was proved there that a semigroup variety \mathcal{V} has a finite degree if and only if it satisfies an identity of the form

$$(1.4) \quad x_1 \cdots x_n = w$$

for some natural n and some word w of length $> n$. Moreover, the proof of this result easily implies that \mathcal{V} has a degree $\leq n$ if and only if it satisfies an identity of the form (1.4) for some word w of length $> n$. For varieties of degree 2, this equational characterization was essentially specified in [3, Lemma 3]. Namely, it was verified there that a semigroup variety has degree ≤ 2 if and only if it satisfies an identity of the form $xy = w$ where w is one of the words $x^{m+1}y$, xy^{m+1} or $(xy)^{m+1}$ for some natural m . In [16, Proposition 2.11], analogous specification of the mentioned above result of [9] was obtained for semigroup

varieties of degree $\leq n$ with arbitrary n (see Proposition 2.9 below). The second objective of this article is to expand the mentioned results of [9, 16] on epigroup varieties. Note that the proof of the corresponding assertion uses Theorem 1.1.

In order to formulate the second result of the article, we need some definitions and notation. A *semigroup word* is a word that does not contain the unary operation. If $w \in F$ then $\ell(w)$ stands for the length of w ; here we assume that the length of any non-semigroup word is infinite. As usual, a pair of identities $wx = xw = w$ where the letter x does not occur in the word w is written as the symbolic identity $w = 0$. Note that this notation is justified because a semigroup with such identities has a zero element and all values of the word w in this semigroup are equal to zero. If the class K_Σ consists of periodic epigroups (in particular, of nilsemigroups) then it is a periodic semigroup variety, and therefore is an epigroup variety. Thus, the notation $\mathbf{V}[\Sigma]$ is correct in this case. We often use this observation below without any additional references. Put

$$\begin{aligned}\mathcal{F} &= \mathbf{V}[x^2 = 0, xy = yx], \\ \mathcal{F}_k &= \mathbf{V}[x^2 = x_1 \cdots x_k = 0, xy = yx]\end{aligned}$$

where k is an arbitrary natural number. The second main result of the article is the following

Theorem 1.2. *For an epigroup variety \mathcal{V} , the following are equivalent:*

- 1) \mathcal{V} is a variety of finite degree;
- 2) $\mathcal{V} \not\subseteq \mathcal{F}$;
- 3) \mathcal{V} satisfies an identity of the form (1.4) for some natural n and some unary word w with $\ell(w) > n$;
- 4) \mathcal{V} satisfies an identity of the form

$$(1.5) \quad x_1 \cdots x_n = x_1 \cdots x_{i-1} \cdot \overline{\overline{x_i \cdots x_j}} \cdot x_{j+1} \cdots x_n$$

for some i, j and n with $1 \leq i \leq j \leq n$.

As we will seen below, the proof of this theorem easily implies the following

Corollary 1.3. *Let n be an arbitrary natural number. For an epigroup variety \mathcal{V} , the following are equivalent:*

- 1) \mathcal{V} is a variety of degree $\leq n$;
- 2) $\mathcal{V} \not\subseteq \mathcal{F}_{n+1}$;
- 3) \mathcal{V} satisfies an identity of the form (1.4) for some unary word w with $\ell(w) > n$;
- 4) \mathcal{V} satisfies an identity of the form (1.5) for some i and j with $1 \leq i \leq j \leq n$.

It is well known that an epigroup variety has degree 1 if and only if it satisfies the identity

$$(1.6) \quad x = \overline{\overline{x}}$$

(see Lemma 2.7 below). Besides that, it is evident that a variety has degree 1 if and only if it does not contain the variety of semigroups with zero multiplication, i. e., the variety \mathcal{F}_2 . The equivalence of the claims 1), 2) and 4) of Corollary 1.3 generalizes these known facts.

The article consists of four sections. Section 2 contains definitions, notation and auxiliary results we need. Section 3 is devoted to the proof of Theorem 1.1, while in Section 4 Theorem 1.2 and Corollary 1.3 are proved.

2. PRELIMINARIES

We denote by $\text{Gr } S$ the set of all group elements of the epigroup S . The following well-known fact was verified in [7].

Lemma 2.1. *If S is an epigroup, $x \in S$ and $x^n \in \text{Gr } S$ for some natural n then $x^m \in \text{Gr } S$ for every $m \geq n$.* \square

In the following lemma, we collect several simple and well known facts (see [10, 11], for instance).

Lemma 2.2. *If S is an epigroup and $x \in S$ then the equalities*

$$(2.1) \quad x \bar{x} = (x \bar{x})^2 = \overline{x \bar{x}},$$

$$(2.2) \quad x \bar{x} = \bar{x} x = x^\omega,$$

$$(2.3) \quad x^\omega x = x x^\omega = \overline{\bar{x}},$$

$$(2.4) \quad \bar{x} = \overline{x^2} x = x \overline{x^2},$$

$$(2.5) \quad \overline{x^n} = \bar{x}^n,$$

$$(2.6) \quad \overline{\overline{\bar{x}}} = \bar{x}$$

hold where n is an arbitrary natural number. \square

The equalities (2.2) show that the expression v^ω is well defined in epigroup identities as a short form of the term $v \bar{v}$. So, the equalities (2.1)–(2.6) are identities valid in arbitrary epigroup. We need the following generalization of the identities (2.2).

Corollary 2.3. *An arbitrary epigroup satisfies the identities*

$$(2.7) \quad x^n \bar{x}^n = \bar{x}^n x^n = x^\omega$$

for any natural number n .

Proof. Let S be an epigroup and $x \in S$. The identities (2.2) and the fact that x^ω is an idempotent in S imply that $x^n \bar{x}^n = \bar{x}^n x^n = (x \bar{x})^n = (x^\omega)^n = x^\omega$. \square

The symbol \equiv denotes the equality relation on F . The number of occurrences of multiplication or unary operation in a word w is called the *weight* of w .

Lemma 2.4. *Let w be a non-semigroup word depending on a letter x only. Then the variety \mathcal{EPI} satisfies an identity*

$$(2.8) \quad w = x^p \bar{x}^q$$

for some $p \geq 0$ and some positive integer q .

Proof. We use induction on the weight of w .

Induction base. If weight of w equals 1 then $w \equiv \bar{x}$ and the identity (2.8) with $p = 0$ and $q = 1$ holds.

Induction step. Suppose that the weight of the word w is $i > 1$. Further considerations are divided into two cases.

Case 1: $w \equiv w_1 w_2$ where weights of the words w_1 and w_2 are lesser than i . Obviously, at least one of the words w_1 or w_2 contains the unary operation. It suffices to consider the case when the word w_1 is non-semigroup. By the induction assumption, the identity $w_1 = x^s \bar{x}^t$ holds in \mathcal{EPI} for some $s \geq 0$ and some positive integer t . If the word w_2 contains the unary operation then, by the induction assumption, the identity $w_2 = x^m \bar{x}^k$ holds in \mathcal{EPI} for some $m \geq 0$ and some $k > 0$. If, otherwise, the word w_2 is a semigroup one then $w_2 \equiv x^r$ for some r . In any case, we may apply the identity (2.2) and conclude that the class \mathcal{EPI} satisfies an identity of the form (2.8).

Case 2: $w \equiv \overline{w_1}$ where the weight of the word w_1 is lesser than i . If the word w_1 is a semigroup one then $w_1 \equiv x^r$ for some r . Taking into account the identity (2.5), we have that the variety \mathcal{EPI} satisfies the identity $w = \bar{x}^r$ here. If, otherwise, the word w_1 contains the unary operation then, by the induction assumption, the identity $w_1 = x^s \bar{x}^t$ holds in \mathcal{EPI} for some $s \geq 0$ and some $t > 0$. If $s > t$ then

$$\begin{aligned}
w \equiv \overline{w_1} &= \overline{x^s \bar{x}^t} \\
&= \overline{x^{s-t} x^t \bar{x}^t} \\
&= \overline{x^{s-t} (x \bar{x})^t} && \text{by (2.2)} \\
&= \overline{x^{s-t} (x \bar{x})^{s-t}} && \text{by (2.1)} \\
&= \overline{x^{s-t} (x^\omega)^{s-t}} && \text{by (2.2)} \\
&= \overline{(xx^\omega)^{s-t}} && \text{by (2.3)} \\
&= \overline{(\bar{x})^{s-t}} && \text{by (2.3)} \\
&= (\bar{x})^{s-t} && \text{by (2.5)} \\
&= \bar{x}^{s-t} && \text{by (2.6)}.
\end{aligned}$$

If $s = t$ then

$$\begin{aligned}
w \equiv \overline{w_1} &= \overline{x^s \bar{x}^s} \\
&= \overline{(x \bar{x})^s} && \text{by (2.2)} \\
&= \overline{x \bar{x}} && \text{by (2.1)} \\
&= x \bar{x} && \text{by (2.1)}.
\end{aligned}$$

Finally, if $s < t$ then

$$\begin{aligned}
w \equiv \overline{w_1} &= \overline{x^s \bar{x}^t} \\
&= \overline{x^s \bar{x}^s \bar{x}^{t-s}} \\
&= \overline{(x \bar{x})^s \bar{x}^{t-s}} && \text{by (2.2)}
\end{aligned}$$

$$\begin{aligned}
 &= \overline{(x \overline{x})^{t-s} \overline{x}^{t-s}} && \text{by (2.1)} \\
 &= \overline{(x \overline{x^2})^{t-s}} && \text{by (2.1)} \\
 &= \overline{(x \overline{x^2})^{t-s}} && \text{by (2.5)} \\
 &= \overline{\overline{x}^{t-s}} && \text{by (2.4)} \\
 &= (\overline{\overline{x}})^{t-s} && \text{by (2.5)} \\
 &= (xx^\omega)^{t-s} && \text{by (2.3)} \\
 &= (x^2 \overline{x})^{t-s} && \text{by (2.2)} \\
 &= x^{2(t-s)} \overline{x}^{t-s} && \text{by (2.2)}.
 \end{aligned}$$

So, we have proved that the variety \mathcal{EPI} satisfies an identity of the form (2.8) in any case. \square

As usual, we say that an epigroup S has an *index* n if $x^n \in \text{Gr } S$ for any $x \in S$ and n is the least number with such a property. Following [10, 11], we denote the class of all epigroups of index $\leq n$ by \mathcal{E}_n . It is well known that \mathcal{E}_n is an epigroup variety (see [10, Proposition 6] or [11, Proposition 2.10], for instance). The identity $u = v$ is said to be a *mixed* identity if exactly one of the words u and v is a semigroup one.

Corollary 2.5. *If a class of unary semigroups K is contained in \mathcal{EPI} and satisfies a mixed identity then K consists of epigroups and $K \subseteq \mathcal{E}_n$ for some n .*

Proof. Suppose that K satisfies a mixed identity $u = v$. Substitute some letter x to all letters occurring in this identity. Then we obtain an identity of the form $x^n = w$ for some positive integer n and some non-semigroup word w depending on the letter x only. According to Lemma 2.4, the variety \mathcal{EPI} satisfies an identity of the form (2.8). Therefore, the class K satisfies the identities

$$\begin{aligned}
 x^n = w &= x^p \overline{x}^q && \text{by (2.8)} \\
 &= (x^p \overline{x}^{q-1}) \overline{x}^2 x && \text{by (2.4)} \\
 &= (x^p \overline{x}^q) x \overline{x} && \text{by (2.2)} \\
 &= x^n x \overline{x} && \text{by (2.8)} \\
 &= x^{n+1} \overline{x}.
 \end{aligned}$$

So, the identity $x^n = x^{n+1} \overline{x}$ holds in the class K . It is well known (see [11, p.334], for instance) that if a unary semigroup $S \in K$ satisfies this identity then S is an epigroup of index $\leq n$. \square

If $w \in F$ then $t(w)$ stands for the last letter of w .

Lemma 2.6. *For any word u , there is a word u^* such that the variety \mathcal{EPI} satisfies the identity $u = u^* z$ where $z \equiv t(u)$.*

Proof. Let u be an arbitrary word and $z \equiv t(u)$. There are two possible cases: either $u \equiv u^* z$ or $u \equiv s_0 \overline{w_1}$ for some (maybe empty) word s_0 . In the second case we apply to u the identity (2.4) and obtain the word $u_1 \equiv s_0 \overline{w_1^2} w_1$ such that the identity $u = u_1$ holds in the variety \mathcal{EPI} . Here we have two possible

cases again: either $w_1 \equiv w^*z$ or $w_1 \equiv s_1 \overline{w_2}$. In the second case we again apply the identity (2.4) and obtain the word $u_2 \equiv s_0 \overline{w_1^2} s_1 \overline{w_2^2} w_2$ such that the identity $u = u_2$ is satisfied by the variety \mathcal{EPI} . One can continue this process. It is clear that after finite number of steps we find a word with the required properties. \square

At the remaining part of the article, u^* has the same sense as in Lemma 2.6.

As we have already mentioned at the beginning of Section 1, every completely regular semigroup is an epigroup. The operation of pseudoinversion on a completely regular semigroup coincides with the operation of taking of the element inverse to a given element x in the maximal subgroup that contains x . The latter operation is the intensively examined unary operation on the class of completely regular semigroups (see the book [8] or Section 6 of the survey [13], for instance). Thus, varieties of completely regular semigroups are epigroup varieties. The following statement is well known.

Lemma 2.7. *For an epigroup variety \mathcal{V} , the following are equivalent:*

- a) \mathcal{V} is completely regular;
- b) \mathcal{V} is a variety of degree 1;
- c) \mathcal{V} satisfies the identity (1.6).

\square

The following claim is evident.

Lemma 2.8. *Every nil-epigroup satisfies the identity $\overline{x} = 0$.*

\square

Now we formulate results about semigroup varieties of finite degree obtained in the articles [9, 16]. An identity $u = v$ is called a *semigroup identity* if both words u and v are semigroup words. For a semigroup variety \mathcal{V} , the following are equivalent:

- a) \mathcal{V} is a variety of finite degree;
- b) $\mathcal{V} \not\subseteq \mathcal{F}$;
- c) \mathcal{V} satisfies an identity of the form (1.4) for some natural n and some semigroup word w with $\ell(w) > n$;
- d) \mathcal{V} satisfies an identity of the form

$$(2.9) \quad x_1 \cdots x_n = x_1 \cdots x_{i-1} \cdot (x_i \cdots x_j)^{m+1} \cdot x_{j+1} \cdots x_n$$

for some m, n, i and j with $1 \leq i \leq j \leq n$.

The equivalence of the claims a)–c) of this statement was proved in [9, Theorem 2], while the equivalence of the claims a) and d) immediately follows from [16, Proposition 2.11]. For varieties of an arbitrary given degree n , the following modification of this assertion is valid.

Proposition 2.9. *Let n be a natural number. For a semigroup variety \mathcal{V} , the following are equivalent:*

- a) \mathcal{V} is a variety of degree $\leq n$;
- b) $\mathcal{V} \not\subseteq \mathcal{F}_{n+1}$;
- c) \mathcal{V} satisfies an identity of the form (1.4) for some semigroup word w with $\ell(w) > n$;

- d) \mathcal{V} satisfies an identity of the form (2.9) for some m, i and j with $1 \leq i \leq j \leq n$. \square

Here the equivalence of the claims a)–c) easily follows from the proof of [9, Theorem 2], while the equivalence of the claims a) and d) is verified in [16, Proposition 2.11].

A semigroup word w is called *linear* if any letter occurs in w at most ones. Recall that an identity of the form

$$x_1 x_2 \cdots x_n = x_{1\pi} x_{2\pi} \cdots x_{n\pi}$$

where π is a non-trivial permutation on the set $\{1, 2, \dots, n\}$ is called *permutational*. If $w \in F$ then $c(w)$ denotes the set of all letters occurring in the word w .

Lemma 2.10. *If an epigroup variety \mathcal{V} satisfies a non-trivial identity of the form (1.4) then either this identity is permutational or \mathcal{V} is a variety of degree $\leq n$.*

Proof. If the word w contains the operation of pseudoinversion then every nilsemigroup in \mathcal{V} satisfies the identity $x_1 \cdots x_n = 0$ by Lemma 2.8. Therefore, \mathcal{V} is a variety of degree $\leq n$ in this case. Thus, we may assume that the word w is a semigroup one. Suppose that $\ell(w) > n$. Then substituting x to all letters occurring in (1.4), we obtain an identity of the form (1.1). Therefore, \mathcal{V} is periodic. Then it may be considered as a variety of semigroups. According to Proposition 2.9, this means that \mathcal{V} is a variety of degree $\leq n$. Suppose now that $\ell(w) \leq n$. If $c(w) \neq \{x_1, \dots, x_n\}$ then $x_i \notin c(w)$ for some $1 \leq i \leq n$. One can substitute x_i^2 for x_i in (1.4). Then we obtain the identity $x_1 \cdots x_{i-1} x_i^2 x_{i+1} \cdots x_n = w$. Put $w' \equiv x_1 \cdots x_{i-1} x_i^2 x_{i+1} \cdots x_n$. Then $x_1 \cdots x_n = w = w'$ holds in \mathcal{V} . Thus, \mathcal{V} satisfies the identity $x_1 \cdots x_n = w'$ and $\ell(w') > n$. As we have seen above, this implies that \mathcal{V} has degree $\leq n$. Finally, if $c(w) = \{x_1, \dots, x_n\}$ then the fact that $\ell(w) \leq n$ implies that $\ell(w) = n$, whence the word w is linear. Therefore, the identity (1.4) is permutational in this case. \square

Put $\mathcal{P} = \mathbf{V}[xy = x^2y, x^2y^2 = y^2x^2]$. We need the following

Lemma 2.11. *If the variety \mathcal{P} satisfies a non-trivial identity of the form (1.4) then $n > 1$ and $w \equiv w'x_n$ for some word w' with $c(w') = \{x_1, \dots, x_{n-1}\}$.*

Proof. It is well known and easy to check that the variety \mathcal{P} is generated by the semigroup

$$P = \langle a, e \mid e^2 = e, ea = a, ae = 0 \rangle = \{e, a, 0\}.$$

This semigroup is finite, whence it is an epigroup. Note that $\bar{e} = e$ and $\bar{a} = 0$. Suppose that $c(w) \neq \{x_1, \dots, x_n\}$. Then there is a letter x that occurs in one part of the identity (1.4) but does not occur in the other one. Substituting 0 to x and e to all other letters occurring in the identity, we obtain the wrong equality $e = 0$. Therefore, $c(w) = \{x_1, \dots, x_n\}$. Substitute now a to x_n and e to all other letters occurring in the identity (1.4). The left part of the equality we obtain equals a . We denote the right part of this equality by b . Thus, P

satisfies the equality $a = b$. If the unary operation applies to the letter x_n in the word w or $t(w) \neq x_n$ then $b = 0$. But $a \neq 0$ in P . Therefore, $w \equiv w'x_n$ for some word w' . If $x_n \in c(w')$ then $b = 0$ again, thus $c(w') = \{x_1, \dots, x_{n-1}\}$. Finally, the word w' is non-empty because the identity (1.4) is non-trivial. Therefore, $n > 1$. \square

Put $\mathcal{C} = \mathbf{V}[x^2 = x^3, xy = yx]$. The unary semigroup variety generated by an epigroup S is denoted by $\text{var } S$. Clearly, if the semigroup S is finite then $\text{var } S$ is a variety of epigroups. The following statement was formulated without proof in [17, Theorem 3.2]¹. We provide the proof here for the sake of completeness.

Proposition 2.12. *Let \mathcal{V} be an epigroup variety. For an arbitrary epigroup $S \in \mathcal{V}$, the set $\text{Gr } S$ is a right ideal in S if and only if the variety \mathcal{V} does not contain the varieties \mathcal{C} and \mathcal{P} .*

Proof. Necessity. It is well known and easy to check that the variety \mathcal{C} is generated by the semigroup

$$C = \langle a, e \mid e^2 = e, ae = ea = a, a^2 = 0 \rangle = \{e, a, 0\}.$$

This semigroup is finite, whence it is an epigroup. Let S be one of the epigroups C and P . Then $\text{Gr } S = \{e, 0\}$ and $ea = a \notin \text{Gr } S$. We see that $\text{Gr } S$ is not a right ideal in S , whence $C, P \notin \mathcal{V}$. Therefore, $\mathcal{C}, \mathcal{P} \not\subseteq \mathcal{V}$.

Sufficiency. Let S be an epigroup in \mathcal{V} such that $\text{Gr } S$ is not a right ideal in S . Then there are elements $x \in \text{Gr } S$ and $y \in S$ with $xy \notin \text{Gr } S$. Put $e = x^\omega$ and $a = xy$. Since $x \in \text{Gr } S$, we have $ex = x$, and therefore $ea = exy = xy = a$. Let A be the subepigroup in S generated by the elements e and a . The equality $ea = a$ implies that every element in A equals to either e or a^k or $a^m e$ for some natural numbers k and m . Let now J be the ideal in A generated by the element ae . Clearly, any element in J equals to either a^k with $k > 1$ or $a^m e$. If $a \notin J$ then the Rees quotient epigroup A/J is isomorphic to the epigroup P . But this is impossible because $\text{var } P = \mathcal{P} \not\subseteq \mathcal{V}$. Therefore, $a \in J$, whence either $a = a^k$ for some $k > 1$ or $a = a^m e$ for some natural m . In the former case we have $a \in \text{Gr } S$, contradicting the choice of the elements x and y . It remains to consider the latter case. Then $ae = (a^m e)e = a^m e^2 = a^m e = a$. Let K be the ideal in A generated by the element a^2 . It is easy to see that every element in K equals to a^k for some $k > 1$. It is clear that $a \notin K$ because $a \in \text{Gr } S$ otherwise. Then the equalities $ea = a$ and $ae = a$ show that the Rees quotient epigroup A/K is isomorphic to the epigroup C . But this is not the case because $\text{var } C = \mathcal{C} \not\subseteq \mathcal{V}$. \square

Note that semigroup varieties with the property that, for any its member S , the set $\text{Gr } S$ is an ideal or right ideal of S were examined in the article [14].

An epigroup variety \mathcal{V} is called a *variety of epigroups with completely regular n th power* if, for any $S \in \mathcal{V}$, the epigroup S^n is completely regular.

¹There is some inaccuracy in the formulation of this assertion in [17]: it contains the words ‘left ideal’ rather than ‘right ideal’.

Lemma 2.13. *An epigroup variety of degree $\leq n$ is a variety of epigroups with completely regular n th power if and only if it does not contain the varieties \mathcal{P} and $\overleftarrow{\mathcal{P}}$.*

Proof. Necessity. Let \mathcal{V} be a variety of epigroups with completely regular n th power. In view of Lemma 2.7 \mathcal{V} satisfies the identity

$$(2.10) \quad x_1 \cdots x_n = \overline{\overline{x_1 \cdots x_n}}.$$

But Lemma 2.11 and the dual statement imply that this identity is false in the varieties \mathcal{P} and $\overleftarrow{\mathcal{P}}$.

Sufficiency. Let \mathcal{V} be a variety of epigroups of degree $\leq n$ that does not contain the varieties \mathcal{P} and $\overleftarrow{\mathcal{P}}$. Further, let $S \in \mathcal{V}$ and $J = \text{Gr } S$. Clearly, the variety \mathcal{C} is not a variety of finite degree, whence $\mathcal{V} \not\subseteq \mathcal{C}$. Thus \mathcal{V} contains none of the varieties \mathcal{C} , \mathcal{P} and $\overleftarrow{\mathcal{P}}$. Now we may apply Proposition 2.12 and the dual statement with the conclusion that J is an ideal in S . If $x \in S$ then $x^n \in J$ for some n . This means that the Rees quotient semigroup S/J is a nilsemigroup. Since \mathcal{V} is a variety of degree $\leq n$, this means that the epigroup S/J satisfies the identity $x_1 x_2 \cdots x_n = 0$. In other words, if $x_1, x_2, \dots, x_n \in S$ then $x_1 x_2 \cdots x_n \in J$. Therefore, $S^n \subseteq J$, whence the epigroup S^n is completely regular. \square

Let Σ be a system of identities written in the language of unary semigroups. As we have already noted, the class K_Σ is not obliged to be a variety. This claim is confirmed by the following

Example 2.14. Put $N_k = \langle a \mid a^{k+1} = 0 \rangle = \{a, a^2, \dots, a^k, 0\}$ for any natural k . The semigroup N_k is finite, therefore it is an epigroup. Put

$$N = \prod_{k \in \mathbb{N}} N_k.$$

Obviously, the semigroup N is not an epigroup because, for example, no power of the element (a, \dots, a, \dots) belongs to a subgroup. Note that the epigroup N_k is commutative for any k . We see that the class K_Σ with $\Sigma = \{xy = yx\}$ is not a variety.

If w is a semigroup word then $\ell_x(w)$ denotes the number of occurrences of the letter x in this word. Recall that a semigroup identity $v = w$ is called *balanced* if $\ell_x(v) = \ell_x(w)$ for any letter x . We call an identity $v = w$ *strictly unary* if v and w are non-semigroup words. We say that an identity $v = w$ *follows from an identity system Σ in the class of all epigroups* (or Σ *implies $v = w$ in the class of all epigroups*) if this identity holds in the class K_Σ . The following statement gives a complete description of identity systems Σ such that K_Σ is a variety.

Proposition 2.15. *Let Σ be a system of identities written in the language of unary semigroups. The following are equivalent:*

- 1) K_Σ is a variety;
- 2) Σ implies in the class of all epigroups some mixed identity;
- 3) Σ contains either a semigroup non-balanced identity or a mixed identity.

Proof. 1) \longrightarrow 3) Suppose that each identity in Σ is either balanced or strictly unary. We note that the epigroup N_k from Example 2.14 satisfies any balanced identity and any strictly unary one. In particular, any identity from Σ holds in the epigroup N_k . Hence $N_k \in K_\Sigma$ for any k . Example 2.14 shows that the class K_Σ is not a variety.

3) \longrightarrow 2) The case when Σ contains a mixed identity is evident. Suppose now that Σ contains a semigroup non-balanced identity $v = w$. Then $\ell_x(u) \neq \ell_x(v)$ for some letter x . If $\ell(u) = \ell(v)$ then we substitute x^2 to x in $u = v$. As a result, we obtain a semigroup non-balanced identity $u' = v'$ such that K satisfies $u' = v'$, $\ell_x(u') \neq \ell_x(v')$ and $\ell(u') \neq \ell(v')$. This allows us to suppose that $\ell(u) \neq \ell(v)$. Substitute some letter x to all letters occurring in this identity. We obtain an identity of the form (1.1). As it was mentioned above, this identity implies in the class of all epigroups the identity (1.2). It remains to note that this identity is mixed.

2) \longrightarrow 1) Obviously, the class K_Σ is closed under taking of subepigroups and homomorphisms. It remains to prove that it is closed under taking of Cartesian products. Let $\{S_i \mid i \in I\}$ be an arbitrary set of epigroups from K_Σ . Consider the semigroup

$$S = \prod_{i \in I} S_i.$$

According to Corollary 2.5, there exists a number n such that $x^n \in \text{Gr } S$ for any $S \in K_\Sigma$ and any $x \in S$. In particular, the epigroup S_i for any $i \in I$ has this property. But then the semigroup S also satisfies this condition, i.e., S is an epigroup. Obviously, any identity from Σ holds in the epigroup S . Therefore, $S \in K_\Sigma$ and we are done. \square

As it was mentioned above, the classes K_Σ and $\text{var}_E \Sigma$ may differ even whenever K_Σ is a variety. This claim is confirmed by the following example that is communicated to the authors by V. Shaprynskiĭ.

Example 2.16. Let $\Sigma = \{x = x^2\}$. Consider the two-element semilattice $T = \{e, 0\}$. We define on T the unary operation $*$ by the rule $e^* = 0^* = 0$. Results of the article [5] imply that the variety \mathcal{EPI} has the following identity basis:

$$(2.11) \quad (xy)z = x(yz), \overline{xy}x = x\overline{yx}, \overline{x^2}x = \overline{x}, x^2\overline{x} = \overline{x}, \overline{xx} = \overline{x}x, \overline{x^p} = \overline{x}^p$$

where p runs over the set of all prime numbers. So, any non-trivial identity from Δ is strictly unary. Therefore, these identities hold in T , whence $T \in \mathcal{EPI} \wedge \text{var } \Sigma = \text{var}_E \Sigma$. But $\overline{e} = e$. Therefore, the unary operation $*$ is not the pseudoinversion on T , thus $T \notin \mathbf{V}[\Sigma]$.

Recall that a semigroup identity $u = v$ is called *homotypical* if $c(u) = c(v)$, and *heterotypical* otherwise. The following claim gives a classification of all identity systems Σ such that $\mathbf{V}[\Sigma] = \text{var}_E \Sigma$.

Lemma 2.17. *Let Σ be a system of identities written in the language of unary semigroups. The following are equivalent:*

- a) $\mathbf{V}[\Sigma] = \text{var}_E \Sigma$;

- b) $\text{var}_E \Sigma$ satisfies a mixed identity;
 c) Σ contains either a semigroup heterotypical identity or a mixed identity.

Proof. a) \longrightarrow c) Suppose that each identity in Σ is either homotypical or strictly unary. Obviously, the unary semigroup T from Example 2.16 satisfies all these identities, whence $T \in \text{var}_E \Sigma$. But $T \notin \mathbf{V}[\Sigma]$, i. e., $\mathbf{V}[\Sigma] \neq \text{var}_E \Sigma$.

c) \longrightarrow b) If the identity $u = v$ is mixed then the required assertion is obvious. Suppose that the identity $u = v$ is heterotypical. We may assume that there is some letter x that occurs in the word u but does not occur in the word v . We substitute \bar{x} to x in $u = v$. As a result, we obtain a mixed identity.

The implication b) \longrightarrow a) follows from Corollary 2.5. \square

Suppose that an identity $u = v$ holds in the variety $\text{var}_E \Sigma$. In view of the generally known universal algebraic considerations, this identity may be obtained from the set of identities $\Sigma \cup \Delta$ by using a finite number of the following operations:

- swap of the left and the right part of the identity,
- equating of two words that are equal to the same word,
- side-by-side multiplication of two identities,
- applying the unary operation to both parts of the identity,
- applying a substitution on F to both parts of the identity.

For convenience of references, we formulate this fact as a lemma.

Lemma 2.18. *Let Σ be a system of identities written in the language of unary semigroups. If an identity $u = v$ holds in the variety $\text{var}_E \Sigma$ then there exist a sequence of identities*

$$(2.12) \quad u_0 = v_0, u_1 = v_1, \dots, u_m = v_m$$

such that the identity $u_0 = v_0$ lies in $\Sigma \cup \Delta$, the identity $u_m = v_m$ coincides with the identity $u = v$ and, for every $i = 1, \dots, m$, one of the following holds:

- (i) *the identity $u_i = v_i$ lies in $\Sigma \cup \Delta$;*
- (ii) *there is $0 \leq j < i$ such that $u_i \equiv v_j$ and $v_i \equiv u_j$;*
- (iii) *there are $0 \leq j, k < i$ such that $u_j \equiv u_i$, $v_j \equiv u_k$ and $v_k \equiv v_i$;*
- (iv) *there are $0 \leq j, k < i$ such that $u_i \equiv u_j u_k$ and $v_i \equiv v_j v_k$;*
- (v) *there is $0 \leq j < i$ such that $u_i \equiv \overline{u_j}$ and $v_i \equiv \overline{v_j}$;*
- (vi) *there is $0 \leq j < i$ such that $u_i \equiv \xi(u_j)$ and $v_i \equiv \xi(v_j)$ for some substitution ξ on F .* \square

Lemma 2.18 immediately implies

Corollary 2.19. *Let a variety $\text{var}_E \Sigma$ satisfies an identity $u = v$. If $t(p) \equiv t(q)$ for any identity $p = q \in \Sigma$ then $t(u) \equiv t(v)$.* \square

The sequence of identities (2.12) with the properties mentioned in Lemma 2.18 is called a *deduction of the identity $v = w$ from the identity system Σ* .

Lemma 2.20. *Let $\Theta = \{p_i = q_i \mid i \in I\}$ be an identity system written in the language of unary semigroups and x a letter. Put $\Sigma = \{p_i x = q_i x \mid i \in I\}$. If an identity $u x = v x$ holds in the variety $\text{var}_E \Sigma$ then the identity $u = v$ holds in the variety $\text{var}_E \Theta$.*

Proof. Let (2.12) be a deduction of the identity $ux = vx$ from the identity system Σ . Let $1 \leq i \leq m$. Corollary 2.19 implies that $t(u_i) \equiv t(v_i)$. We are going to verify that the variety $\text{var}_E \Sigma$ satisfies the identity $u_i^* = v_i^*$. Then, in particular, $\text{var}_E \Sigma$ satisfies $u = v$. We will use induction by i . It will be convenient for us to suppose that Δ is an identity basis of the variety \mathcal{EPI} rather than the set of all identities that hold in this variety. It is clear that, under this assumption, all considerations valid. Thus, we assume that Δ coincides with the identity system (2.11).

Induction base. If $u_0 = v_0 \in \Sigma$ then the statement we prove is evident. If $u_0 = v_0 \in \Delta$ then it may be verified easily that the identity $u_0^* = v_0^*$ holds in the variety \mathcal{EPI} .

Induction step. Let now $i > 0$. We need consider the cases (i)–(vi) listed in Lemma 2.18.

(i) This case is proved analogously to induction base.

(ii) The identity $u_i^* = v_j^*$ holds in the variety $\text{var}_E \Theta$ by induction assumption. Since the identities $u_i^* = v_i^*$ and $v_j^* = u_j^*$ coincide, we are done.

(iii) The identities $u_j^* = v_j^*$ and $u_k^* = v_k^*$ (i. e., $u_i^* = u_k^*$ and $u_k^* = v_i^*$, respectively) hold in $\text{var}_E \Theta$ by induction assumption. Therefore, the identity $u_i^* = v_i^*$ holds in $\text{var}_E \Theta$ as well.

(iv) Note that $u_i^* \equiv u_j u_k^*$ and $v_i^* \equiv v_j v_k^*$. Let $y \equiv t(u_j) \equiv t(v_j)$. Using the inductive assumption, we conclude that the identities $u_j^* = v_j^*$ and $u_k^* = v_k^*$ hold in $\text{var}_E \Theta$. Multiplying the former identity on a letter y on the right, we see that $\text{var}_E \Theta$ satisfies the identity $u_j^* y = v_j^* y$. Since the variety \mathcal{EPI} satisfies the identities $u_j = u_j^* y$, $v_j = v_j^* y$ and $\text{var}_E \Theta \subseteq \mathcal{EPI}$, we have that $\text{var}_E \Theta$ satisfies the identity $u_j = v_j$. Multiplying this identity and the identity $u_k^* = v_k^*$, we conclude that $\text{var}_E \Theta$ satisfies the identity $u_i^* = v_i^*$.

(v) The identity $u_j^* = v_j^*$ holds in the variety $\text{var}_E \Theta$ by inductive assumption. Note that $u_i^* \equiv \overline{u_j}^2 u_j^*$ and $v_i^* \equiv \overline{v_j}^2 v_j^*$. As we have seen in the case (iv), $\text{var}_E \Theta$ satisfies the identity $u_j = v_j$. It is evident that the identity $\overline{u_j}^2 u_j^* = \overline{v_j}^2 v_j^*$ may be deduced from the identities $u_j = v_j$ and $u_j^* = v_j^*$. Whence $\text{var}_E \Theta$ satisfies the identity $u_i^* = v_i^*$.

(vi) As usual, $u_j^* = v_j^*$ holds in $\text{var}_E \Theta$ by inductive assumption. Let $t(u_j) \equiv t(v_j) \equiv x$. Then $(\xi(u_j))^* \equiv \xi(u_j^*)(\xi(x))^*$ and $(\xi(v_j))^* \equiv \xi(v_j^*)(\xi(x))^*$. This implies that $\text{var}_E \Theta$ satisfies the identity $u_i^* = v_i^*$. \square

3. ENDOMORPHISMS OF THE LATTICE EPI

To verify Theorem 1.1, we need some auxiliary facts.

Lemma 3.1. *Let $\Sigma = \{p_\alpha = q_\alpha \mid \alpha \in \Lambda\}$. If the variety $\text{var}_E \Sigma$ satisfies an identity $u = v$ and x is a letter that does not occur in the words u, v, p_α and q_α (for all $\alpha \in \Lambda$) then the identity $ux = vx$ follows from the identity system $\Sigma' = \{p_\alpha x = q_\alpha x \mid \alpha \in \Lambda\}$ in the class of all epigroups.*

Proof. Let (2.12) be a deduction of the identity $u = v$ from the identity system Σ . Let $1 \leq i \leq m$. Corollary 2.19 implies that $t(u_i) \equiv t(v_i)$. Let y be a letter

with $y \neq x$. If the letter x occurs in some identities of the sequence (2.12) then we substitute y to x in all such identities. The identities from $\Sigma \cup \{u = v\}$ will not change because these identities do not contain the letter x ; and the identities from Δ will still remain in Δ . The sequence we obtain is a deduction of the identity $u = v$ from the identity system $\Sigma \cup \Delta$ again, and all the identities of this deduction do not contain the letter x . We may assume without any loss that already the deduction (2.12) possesses the last property.

For each $i = 0, 1, \dots, m$, the identity $u_i = v_i$ holds in the variety $\text{var}_E \Sigma$. Since the identity $u_m = v_m$ coincides with the identity $u = v$, it suffices to verify that, for each $i = 0, 1, \dots, m$, the identity $u_i x = v_i x$ follows from the identity system Σ' in the class of all epigroups. The proof of this claim is given by induction on i .

Induction base is evident because the identity $u_0 = v_0$ lies in $\Sigma \cup \Delta$.

Induction step. Let now $i > 0$. One can consider the cases (i)–(vi).

(i) This case is obvious.

(ii) By the induction assumption, the identity $u_j x = v_j x$ follows from the identity system Σ' in the class of all epigroups. Since the identity $u_i x = v_i x$ coincides with the identity $v_j x = u_j x$, we are done.

(iii) By the induction assumption, the identities $u_j x = v_j x$ (i. e., $u_i x = u_k x$) and $u_k x = v_k x$ (i. e., $u_k x = v_i x$) follow from the identity system Σ' in the class of all epigroups. Therefore, the identity $u_i x = v_i x$ follows from the identity system Σ' in the class of all epigroups too.

(iv) By the induction assumption, the identities $u_j x = v_j x$ and $u_k x = v_k x$ follow from the identity system Σ' in the class of all epigroups. We substitute $u_k x$ to x in the identity $u_j x = v_j x$. Since the letter x does not occur in the words u_j and v_j , we obtain the identity $u_j u_k x = v_j u_k x$, i. e., $u_i x = v_j u_k x$. Further, we multiply the identity $u_k x = v_k x$ on v_j from the left. Here we obtain the identity $v_j u_k x = v_j v_k x$, i. e., $v_j u_k x = v_i x$. We see that the identity system Σ' implies the identities $u_i x = v_j u_k x$ and $v_j u_k x = v_i x$ in the class of all epigroups, whence the identity $u_i x = v_i x$ also follows from Σ' in the class of all epigroups.

(v) By the induction assumption, the identity $u_j x = v_j x$ follows from the identity system Σ' in the class of all epigroups. Since $u_i \equiv \overline{u_j}$ and $v_i \equiv \overline{v_j}$, it remains to verify that the identity $\overline{u_j} x = \overline{v_j} x$ follows from the identity system Σ' in the class of all epigroups. Suppose that an epigroup S satisfies the identity $u_j x = v_j x$ and $|c(u_j) \cup c(v_j)| = k$. We fix arbitrary elements a_1, \dots, a_k and b in S . Put $U_j = u_j(a_1, \dots, a_k)$ and $V_j = v_j(a_1, \dots, a_k)$. Then

$$(3.1) \quad U_j b = V_j b.$$

We need to verify that $\overline{U_j} b = \overline{V_j} b$. First of all, we verify that

$$(3.2) \quad V_j^{s+1} = U_j^s V_j$$

for any natural s . We use induction by s . If $s = 1$ then the equality (3.2) coincides with (3.1) where $b = V_j$. If $s > 1$ then

$$\begin{aligned}
 V_j^{s+1} &= V_j V_j^s \\
 &= U_j V_j^s && \text{by (3.1) with } b = V_j^s \\
 &= U_j U_j^{s-1} V_j && \text{by the inductive assumption} \\
 &= U_j^s V_j,
 \end{aligned}$$

and the equality (3.2) is proved. The equality (3.1) with $b = V_j$ and (2.2) imply that $U_j^\omega V_j = \overline{U_j} U_j V_j = \overline{U_j} V_j^2$. Thus,

$$(3.3) \quad \overline{U_j} V_j^2 = U_j^\omega V_j.$$

Let now s be a natural number with $s \geq 2$. Using (3.3), we have

$$\overline{U_j}^s V_j^s = \overline{U_j}^{s-1} (\overline{U_j} V_j^2) V_j^{s-2} = \overline{U_j}^{s-1} U_j^\omega V_j V_j^{s-2} = \overline{U_j}^{s-1} V_j^{s-1}.$$

Therefore, $\overline{U_j}^s V_j^s = \overline{U_j}^{s-1} V_j^{s-1} = \dots = \overline{U_j} V_j$. Thus,

$$(3.4) \quad \overline{U_j}^s V_j^s = \overline{U_j} V_j$$

for any natural s . Since S is an epigroup, there are numbers g and h such that $U_j^g, V_j^h \in \text{Gr } S$. Put $m = \max\{g, h\}$. For any $s \geq m$ we have

$$\begin{aligned}
 U_j^\omega V_j^s &= U_j^\omega (V_j^s V_j^\omega) && \text{because } V_j^s \in G_{V_j} \text{ by Lemma 2.1} \\
 &= U_j^\omega (V_j^{s+1} \overline{V_j}) && \text{by (2.2)} \\
 &= (U_j^\omega U_j^s) V_j \overline{V_j} && \text{by (3.2)} \\
 &= (U_j^s V_j) \overline{V_j} && \text{because } U_j^s \in G_{U_j} \text{ by Lemma 2.1} \\
 &= V_j^{s+1} \overline{V_j} && \text{by (3.2)} \\
 &= V_j^s V_j^\omega && \text{by (2.2)} \\
 &= V_j^s && \text{because } V_j^s \in G_{V_j} \text{ by Lemma 2.1.}
 \end{aligned}$$

Thus,

$$(3.5) \quad U_j^\omega V_j^s = V_j^s$$

for any $s \geq m$. Note also that

$$\begin{aligned}
 U_j^\omega V_j &= \overline{U_j}^m U_j^m V_j && \text{by (2.7)} \\
 &= \overline{U_j}^m (U_j^m V_j) && \text{by (2.5)} \\
 &= \overline{U_j}^m V_j^{m+1} && \text{by (3.2)} \\
 &= \overline{U_j}^m V_j^{m+1} V_j^\omega && \text{because } V_j^{m+1} \in G_{V_j} \text{ by Lemma 2.1} \\
 &= \overline{U_j}^m V_j^{m+1} (V_j^m)^\omega && \text{because } G_{V_j} = G_{V_j^m} \\
 &= \overline{U_j}^m V_j^{m+1} V_j^m \overline{V_j^m} && \text{by (2.2)} \\
 &= (\overline{U_j}^m V_j^m) V_j^{m+1} \overline{V_j^m} && \text{by (2.5)} \\
 &= \overline{U_j} V_j V_j^{m+1} \overline{V_j^m} && \text{by (3.4)}
 \end{aligned}$$

$$\begin{aligned}
 &= (\overline{U_j} V_j^2)(V_j^m \overline{V_j^m}) \\
 &= (U_j^\omega V_j)(V_j^m \overline{V_j^m}) && \text{by (3.3)} \\
 &= V_j^{m+1} \overline{V_j^m} && \text{by (3.5)} \\
 &= V_j V_j^m \overline{V_j^m} && \text{by (2.5)} \\
 &= V_j V_j^\omega && \text{by (2.7)} \\
 &= \overline{\overline{V_j}} && \text{by (2.3).}
 \end{aligned}$$

Thus,

$$(3.6) \quad U_j^\omega V_j = \overline{\overline{V_j}}.$$

Besides that,

$$\begin{aligned}
 \overline{U_j} V_j^\omega &= (\overline{U_j} V_j^2) \overline{V_j^2} && \text{by (2.7)} \\
 &= (U_j^\omega V_j) \overline{V_j^2} && \text{by (3.3)} \\
 &= \overline{\overline{V_j}} \overline{V_j^2} && \text{by (3.6)} \\
 &= V_j^\omega \overline{V_j} && \text{because } \overline{\overline{V_j}} \text{ and } \overline{V_j} \text{ are mutually inverse in } G_{V_j} \\
 &= \overline{V_j} && \text{because } \overline{V_j} \in G_{V_j}.
 \end{aligned}$$

Thus,

$$(3.7) \quad \overline{U_j} V_j^\omega = \overline{V_j}.$$

Finally, we have

$$\begin{aligned}
 \overline{U_j} b &= \overline{U_j^2} (U_j b) && \text{by (2.4)} \\
 &= \overline{U_j^2} (V_j b) && \text{by (3.1)} \\
 &= \overline{U_j^2} (U_j^\omega V_j) b && \text{because } \overline{U_j^2} \in G_{U_j} \\
 &= \overline{U_j^2} (\overline{\overline{V_j}} b) && \text{by (3.6)} \\
 &= (\overline{U_j^2} V_j^\omega) V_j b && \text{by (2.3)} \\
 &= \overline{U_j}^2 V_j^\omega V_j b && \text{by (2.5)} \\
 &= \overline{U_j} \overline{V_j} V_j b && \text{by (3.7)} \\
 &= \overline{U_j} V_j^\omega \overline{V_j} V_j b && \text{because } \overline{V_j} \in G_{V_j} \\
 &= \overline{U_j} V_j^\omega \overline{V_j} V_j b && \text{because } \overline{V_j} \in G_{V_j} \\
 &= \overline{V_j} (\overline{V_j} V_j) b && \text{by (3.7)} \\
 &= \overline{V_j} V_j^\omega b && \text{by (2.2)} \\
 &= \overline{V_j} b && \text{because } \overline{V_j} \in G_{V_j}.
 \end{aligned}$$

We prove that $\overline{U_j} b = \overline{V_j} b$. This completes a consideration of the case (v).

(vi) By the induction assumption, the identity $u_j x = v_j x$ follows from the identity system Σ' in the class of all epigroups. We may assume without any loss that $c(u_j) \cup c(v_j) = \{x_1, \dots, x_k\}$ and the identity $u_i = v_i$ is obtained from

the identity $u_j = v_j$ by a substitution of some word w for some letter that occurs in the identity $u_j = v_j$. Since $x \notin c(u_i) \cup c(v_i)$, the letter x does not occur in the word w . We substitute w to x in the identity $u_j x = v_j x$. Then we obtain the identity $u_i x = v_i x$. Therefore, this identity follows from the identity system Σ' in the class of all epigroups. \square

Lemma 3.2. *Let $\Theta = \{p_i = q_i \mid i \in I\}$ be an identity system such that the variety $\text{var}_E \Theta$ is a variety of epigroups, and let x be a letter with $x \notin c(p_i) \cup c(q_i)$ for all $i \in I$. Put $\Sigma = \{p_i x = q_i x \mid i \in I\}$. Then the variety $\text{var}_E \Sigma$ is a variety of epigroups too, and the identity $ux = vx$ holds in $\text{var}_E \Sigma$ whenever the identity $u = v$ holds in $\text{var}_E \Theta$.*

Proof. Let y be a letter with $y \notin c(u) \cup c(v)$ and $y \notin c(p_i) \cup c(q_i)$ for all $i \in I$. According to Lemma 3.1, the identity $uy = vy$ holds in the class of epigroups K_Σ . Substituting x to y in this identity, we obtain that the identity $ux = vx$ is valid in K_Σ . Now Lemma 2.17 applies with the conclusion that the system Θ contains an identity $p_i = q_i$ that is either a mixed identity or a semigroup heterotypical one. Then the identity $p_i x = q_i x \in \Sigma$ also has one of these two properties. In view of Proposition 2.15, the class K_Σ is an epigroup variety. Now Lemma 2.17 applies again, and we conclude that $K_\Sigma = V[\Sigma] = \text{var}_E \Sigma$. \square

Corollary 3.3. *Let $\Theta_1 = \{v_i = w_i \mid i \in I\}$, $\Theta_2 = \{p_\alpha = q_\alpha \mid \alpha \in \Lambda\}$ and x a letter such that $x \notin c(v_i) \cup c(w_i)$ for all $i \in I$ and $x \notin c(p_\alpha) \cup c(q_\alpha)$ for all $\alpha \in \Lambda$. Put $\Sigma_1 = \{v_i x = w_i x \mid i \in I\}$ and $\Sigma_2 = \{p_\alpha x = q_\alpha x \mid \alpha \in \Lambda\}$. If $\text{var}_E \Theta_1$ is an epigroup variety then $\text{var}_E \Theta_1 = \text{var}_E \Theta_2$ if and only if $\text{var}_E \Sigma_1 = \text{var}_E \Sigma_2$.*

Proof. Necessity. Let $\text{var}_E \Theta_1 = \text{var}_E \Theta_2$. This means that the variety $\text{var}_E \Theta_1$ satisfies the identity $p_\alpha = q_\alpha$ for each $\alpha \in \Lambda$. Lemma 3.2 implies that the identity $p_\alpha x = q_\alpha x$ holds in the variety $\text{var}_E \Sigma_1$. Therefore, $\text{var}_E \Sigma_1 \subseteq \text{var}_E \Sigma_2$. The opposite inclusion holds as well by symmetry. Thus $\text{var}_E \Sigma_1 = \text{var}_E \Sigma_2$.

Sufficiency. Let now $\text{var}_E \Sigma_1 = \text{var}_E \Sigma_2$. Then $p_\alpha x = q_\alpha x$ holds in $\text{var}_E \Sigma_1$ for each $\alpha \in \Lambda$. Lemma 2.20 then implies that the variety $\text{var}_E \Theta_1$ satisfies all identities $p_\alpha = q_\alpha$, whence $\text{var}_E \Theta_1 \subseteq \text{var}_E \Theta_2$. By symmetry, $\text{var}_E \Theta_2 \subseteq \text{var}_E \Theta_1$. Therefore, $\text{var}_E \Theta_1 = \text{var}_E \Theta_2$. \square

Now we are well prepared to complete the proof of Theorem 1.1. By symmetry and evident induction, it suffices to verify that if x is a letter with $x \notin c(u_i) \cup c(v_i)$ for all $i \in I$ then $\mathcal{V}^{0,1} = \text{var}_E \{u_i x = v_i x \mid i \in I\}$ is an epigroup variety and the map f from **EPI** into itself given by the rule $f(\mathcal{V}) = \mathcal{V}^{0,1}$ is an injective endomorphism of the lattice **EPI**.

The claim that $\mathcal{V}^{0,1}$ is an epigroup variety follows from Lemma 3.2. The map f is well defined and injective by Corollary 3.3. It remains to verify that the map f is a homomorphism. Let \mathcal{V}_1 and \mathcal{V}_2 be epigroup varieties. Further, let $\mathcal{V}_1 = \text{var}_E \Theta_1$ and $\mathcal{V}_2 = \text{var}_E \Theta_2$ for identity systems $\Theta_1 = \{u_i = v_i \mid i \in I\}$ and $\Theta_2 = \{p_\alpha = q_\alpha \mid \alpha \in \Lambda\}$. Suppose that x is a letter such that $x \notin c(u_i) \cup c(v_i)$ for all $i \in I$ and $x \notin c(p_\alpha) \cup c(q_\alpha)$ for all $\alpha \in \Lambda$. Put $\Sigma_1 = \{u_i x = v_i x \mid i \in I\}$ and $\Sigma_2 = \{p_\alpha x = q_\alpha x \mid \alpha \in \Lambda\}$. Then we have

$$\begin{aligned} f(\mathcal{V}_1 \wedge \mathcal{V}_2) &= f(\text{var}_E \Theta_1 \wedge \text{var}_E \Theta_2) = f(\text{var}_E(\Theta_1 \cup \Theta_2)) \\ &= \text{var}_E(\Sigma_1 \cup \Sigma_2) = \text{var}_E \Sigma_1 \wedge \text{var}_E \Sigma_2 = f(\mathcal{V}_1) \wedge f(\mathcal{V}_2). \end{aligned}$$

Thus, $f(\mathcal{V}_1 \wedge \mathcal{V}_2) = f(\mathcal{V}_1) \wedge f(\mathcal{V}_2)$. It remains to verify that $f(\mathcal{V}_1 \vee \mathcal{V}_2) = f(\mathcal{V}_1) \vee f(\mathcal{V}_2)$.

Let $\mathcal{V}_1 \vee \mathcal{V}_2 = \text{var}_E \Theta$ where $\Theta = \{s_j = t_j \mid j \in J\}$. We may assume without loss of generality that $x \notin c(s_j) \cup c(t_j)$ for all $j \in J$ because we may rename letters otherwise. Further, $\text{var}_E \Theta_1 = \text{var}_E(\Theta_1 \cup \Theta)$ because $\mathcal{V}_1 \subseteq \mathcal{V}_1 \vee \mathcal{V}_2$. Now Corollary 3.3 applies with the conclusion that

$$\begin{aligned} f(\mathcal{V}_1) &= f(\text{var}_E(\Theta_1 \cup \Theta)) = \text{var}_E\{u_i x = v_i x, s_j x = t_j x \mid i \in I, j \in J\} \\ &\subseteq \text{var}_E\{s_j x = t_j x \mid j \in J\} = f(\mathcal{V}_1 \vee \mathcal{V}_2). \end{aligned}$$

Analogously, $f(\mathcal{V}_2) \subseteq f(\mathcal{V}_1 \vee \mathcal{V}_2)$. Therefore, $f(\mathcal{V}_1) \vee f(\mathcal{V}_2) \subseteq f(\mathcal{V}_1 \vee \mathcal{V}_2)$.

It remains to verify the opposite inclusion. Let the identity $u = v$ holds in the variety $f(\mathcal{V}_1) \vee f(\mathcal{V}_2)$. Then it holds in $f(\mathcal{V}_i)$ with $i = 1, 2$. We may assume without any loss that $t(u) \equiv x$. Applying Corollary 2.19, we conclude that $t(v) \equiv t(u) \equiv x$. Lemma 2.6 implies that the identities $u = u^*x$ and $v = v^*x$ hold in the variety \mathcal{EPI} . Whence, the variety $f(\mathcal{V}_1)$ satisfies the identity $u^*x = v^*x$. Now Lemma 2.20 applies and we conclude that the variety \mathcal{V}_1 satisfies the identity $u^* = v^*$. Analogous considerations show that this identity is true in the variety \mathcal{V}_2 as well. Thus, $u^* = v^*$ holds in the variety $\mathcal{V}_1 \vee \mathcal{V}_2$. The letter x does not occur in any identity from Θ . Now Lemma 3.2 applies with the conclusion that the variety $f(\mathcal{V}_1 \vee \mathcal{V}_2)$ satisfies the identity $u^*x = v^*x$. Then this variety satisfies the identity $u = v$ too. Therefore, $f(\mathcal{V}_1 \vee \mathcal{V}_2) \subseteq f(\mathcal{V}_1) \vee f(\mathcal{V}_2)$. Theorem 1.1 is proved. \square

4. VARIETIES OF FINITE DEGREE

The aim of this section is to prove Theorem 1.2 and Corollary 1.3. The implication 4) \longrightarrow 3) of Theorem 1.2 is obvious, while the implication 3) \longrightarrow 2) follows from the evident fact that the variety \mathcal{F} does not satisfy an identity of the form (1.4) with $\ell(w) > n$. It remains to verify the implications 1) \longrightarrow 4) and 2) \longrightarrow 1).

1) \longrightarrow 4) We are going to verify that if an epigroup variety \mathcal{V} is a variety of degree $\leq n$ then it satisfies an identity of the form (1.5) for some i and j with $1 \leq i \leq j \leq n$. Clearly, this implies the implication. We use induction by n .

Induction base. If \mathcal{V} is a variety of degree 1 then it satisfies the identity of the form (1.5) with $i = j = n = 1$ by Lemma 2.7.

Induction step. Let $n > 1$ and \mathcal{V} is a variety of degree $\leq n$. If $\mathcal{P}, \overleftarrow{\mathcal{P}} \not\subseteq \mathcal{V}$ then \mathcal{V} is a variety of epigroups with completely regular n th power by Lemma 2.13. By Lemma 2.7 \mathcal{V} then satisfies the identity (2.10), i.e., the identity of the form (1.5) with $i = 1$ and $j = n$. Suppose now that \mathcal{V} contains one of the varieties \mathcal{P} or $\overleftarrow{\mathcal{P}}$. We will assume without loss of generality that $\mathcal{P} \subseteq \mathcal{V}$.

The variety \mathcal{F}_{n+1} has degree $n+1$, whence $\mathcal{V} \not\subseteq \mathcal{F}_{n+1}$. Therefore, there is an identity $u = v$ that holds in \mathcal{V} but is false in \mathcal{F}_{n+1} . In view of Lemma 2.8, every non-semigroup word equals to 0 in \mathcal{F}_{n+1} . It is evident that every non-linear semigroup word and every semigroup word of length $> n$ equal to 0 in \mathcal{F}_{n+1} as well. Therefore, we may assume without any loss that u is a linear semigroup word of length $\leq n$, i.e., $u \equiv x_1 \cdots x_m$ for some $m \leq n$. Since $\mathcal{P} \subseteq \mathcal{V}$, the identity $x_1 \cdots x_m = v$ holds in \mathcal{P} . Now Lemma 2.11 applies with the conclusion

that $m > 1$ and $v \equiv v'x_m$ for some word v' with $c(v') = \{x_1, \dots, x_{m-1}\}$. Suppose that $\ell(v') \leq m-1$. In particular, this means that v' is a semigroup word. Since $c(v') = \{x_1, \dots, x_{m-1}\}$, we have that $\ell(v') = m-1$. Therefore, the word v' is linear, whence v is linear too. This means that $u = v$ is a permutational identity. But every permutational identity holds in the variety \mathcal{F}_{n+1} , while the identity $u = v$ is false in \mathcal{F}_{n+1} . Hence $\ell(v') > m-1$. So, taking into account Proposition 2.15, we have that $\mathcal{V} \subseteq \mathbf{V}[x_1 \cdots x_m = v'x_m]$.

Also, Proposition 2.15 implies that the class of epigroups satisfying the identity

$$(4.1) \quad x_1 \cdots x_{m-1} = v'$$

is a variety. We denote this variety by \mathcal{V}' . According to Lemma 2.10, \mathcal{V}' is a variety of degree $\leq m-1$. Since $m \leq n$, we use inductive assumption and conclude that

$$\mathcal{V}' \subseteq \text{var}_E\{x_1 \cdots x_{m-1} = x_1 \cdots x_{i-1} \cdot \overline{x_i \cdots x_j} \cdot x_{j+1} \cdots x_{m-1}\}$$

for some $1 \leq i \leq j \leq m-1$. Further considerations are divided into two cases.

Case 1: the word v' contains the unary operation. According to Lemma 2.17, $\mathcal{V}' = \mathbf{V}[x_1 \cdots x_{m-1} = v'] = \text{var}_E\{x_1 \cdots x_{m-1} = v'\}$. The letter x_m does not occur in any of the words $x_1 \cdots x_{m-1}$, v' and $x_1 \cdots x_{i-1} \cdot \overline{x_i \cdots x_j} \cdot x_{j+1} \cdots x_{m-1}$. Now Theorem 1.1 applies with the conclusion that

$$\text{var}_E\{x_1 \cdots x_m = v'x_m\} \subseteq \text{var}_E\{x_1 \cdots x_m = x_1 \cdots x_{i-1} \cdot \overline{x_i \cdots x_j} \cdot x_{j+1} \cdots x_m\}.$$

Therefore, \mathcal{V} satisfies the identity

$$(4.2) \quad x_1 \cdots x_m = x_1 \cdots x_{i-1} \cdot \overline{x_i \cdots x_j} \cdot x_{j+1} \cdots x_m.$$

It is evident that this identity implies the identity (1.5).

Case 2: w' is a semigroup word. Substitute some letter x to all letters occurring in the identity (4.1). Then we obtain an identity $x^{m-1} = x^{m-1+k}$ for some $k > 0$. By (1.2), the latter identity implies in the class of all epigroups the identity $\bar{x} = x^{mk-1}$. Using Lemma 2.17 we have

$$\begin{aligned} \mathcal{V}' &= \mathbf{V}[x_1 \cdots x_{m-1} = v'] = \mathbf{V}[x_1 \cdots x_{m-1} = v', \bar{x} = x^{mk-1}] \\ &= \text{var}_E\{x_1 \cdots x_{m-1} = v', \bar{x} = x^{mk-1}\}. \end{aligned}$$

As in the Case 1, we apply Theorem 1.1. We get that the variety

$$\text{var}_E\{x_1 \cdots x_m = x_1 \cdots x_{i-1} \cdot \overline{x_i \cdots x_j} \cdot x_{j+1} \cdots x_m\}.$$

contains the variety $\text{var}_E\{x_1 \cdots x_m = v'x_m, \bar{x}x_m = x^{mk-1}x_m\}$. Note that the variety $\mathbf{V}[x_1 \cdots x_m = v'x_m]$ satisfies the identity $x^m = x^{m+k}$. Hence, taking into account (1.3), we have that the identity $\bar{x} = x^{mk-1}$ holds in this variety. Then the variety $\mathbf{V}[x_1 \cdots x_m = v'x_m]$ satisfies the identity $\bar{x}x_m = x^{mk-1}x_m$. Whence, the identity (4.2) holds in the variety $\mathbf{V}[x_1 \cdots x_m = v'x_m]$. Then this variety satisfies the identity (1.5). Thus, we complete the proof of the implication 1) \longrightarrow 4).

2) \longrightarrow 1) Let $\mathcal{V} \not\subseteq \mathcal{F}$. Then there is an identity $u = v$ that holds in \mathcal{V} but does not hold in \mathcal{F} . Repeating literally arguments from the proof of the implication 1) \longrightarrow 4), we reduce our consideration to the case when the word u is linear.

Now Lemma 2.10 and the fact that every permutational identity holds in the variety \mathcal{F} imply that \mathcal{V} is a variety of finite degree. Theorem 1.2 is proved. \square

It remains to prove Corollary 1.3. The implication 1) \longrightarrow 4) of this corollary follows from the proof of the same implication in Theorem 1.2. The implication 4) \longrightarrow 3) is evident, while the implication 3) \longrightarrow 2) follows from the evident fact that the variety \mathcal{F}_{n+1} does not satisfy an identity of the form (1.4) with $\ell(w) > n$. Finally, the implication 2) \longrightarrow 1) of Corollary 1.3 is verified quite analogously to the same implication of Theorem 1.2. \square

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